楕円関数とおもしろい応用

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2015.9.22

三角関数

Definition.

"円周率" π を

$$\frac{\pi}{2} := \int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

で定める．また,

$$x = \sin u \overset{\text{def.}}{\iff} u = \int_0^x \frac{dx}{\sqrt{1-x^2}} \quad \left(\begin{array}{c} -1 \leq x \leq 1 \\ -\pi/2 \leq u \leq \pi/2 \end{array} \right)$$

とする．

$x^2 + y^2 = 1$ の微分を考えて

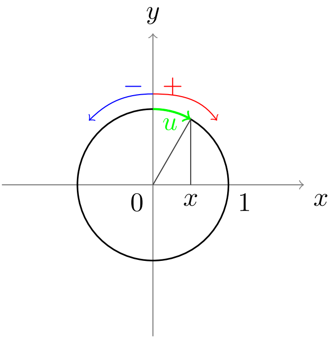
$$x + y\frac{dy}{dx} = 0, \text{よって} \frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow u = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= \int_0^x \sqrt{\frac{y^2 + x^2}{y^2}} dx$$

$$= \int_0^x \frac{dx}{y}$$

$$= \int_0^x \frac{dx}{\sqrt{1 - x^2}}.$$



$\rightsquigarrow$ 幾何学的に $\mathbb{R}$ 全体に拡張する.

Proposition.

$$\sin(u + 2\pi) = \sin u,$$
$$\sin(-u) = -\sin u.$$

Proof.

後半のみ示す.

$$\int_0^{-y_0} \frac{dx}{\sqrt{1-x^2}} \underset{(x=-y)}{=} -\int_0^{y_0} \frac{dy}{\sqrt{1-y^2}} =: -u$$
$$\therefore \sin(-u) = -y_0 = -\sin u.$$

□

Theorem.

u, v を十分小さくとれば次式が成立；

$$\sin(u+v) = \sin u\sqrt{1-\sin^2 v} + \sin v\sqrt{1-\sin^2 u}$$

Proof.

$z = x\sqrt{1-y^2} + y\sqrt{1-x^2}$ とおく． z :十分小を fix. ここで

$$\begin{array}{c|ccc} x & 0 & \to & x \\ y & z & \to & y \end{array}$$

に注意.

$$dz = \left(\sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}}\right)dx + \left(-\frac{xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}\right)dy = 0$$

$$\therefore \left(\sqrt{(1-x^2)(1-y^2)} - xy\right)\left(\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}}\right) = 0.$$

Proof (Cont.)

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} = 0$$

$$\Rightarrow \int_0^x \frac{dx}{\sqrt{1-x^2}} + \int_z^y \frac{dy}{\sqrt{1-y^2}} = 0$$

$$\Rightarrow \underbrace{\int_0^x \frac{dx}{\sqrt{1-x^2}}}_{=:\,u} + \underbrace{\int_0^y \frac{dy}{\sqrt{1-y^2}}}_{=:\,v} = \int_0^z \frac{dz}{\sqrt{1-z^2}}.$$

すなわち次が成り立つ.

$$\sin(u+v) = z = x\sqrt{1-y^2} + y\sqrt{1-x^2}.$$

ここで $x = \sin u,\ y = \sin v$ より証明は完了する. □

$\cos$ 関数を次のようにして定める.

Definition.

$$\cos u := \sin\left(\frac{\pi}{2} - u\right).$$

ここで次は明らか.

Proposition.

$$\cos^2 u + \sin^2 u = 1.$$

Theorem.

$$\sin(u+v) = \sin u \cos v + \cos u \sin v.$$

Question.

$\sin\dfrac{\pi}{6}$ を求めよ.

レムニスケート関数

Definition.

“レムニスケート周率” ϖ を

$$\frac{\varpi}{2} := \int_0^1 \frac{dx}{\sqrt{1-x^4}}$$

で定める．また，

$$x = \operatorname{sl} u \overset{\text{def.}}{\iff} u = \int_0^x \frac{dx}{\sqrt{1-x^4}} \quad \left(\begin{array}{c} -1 \leq x \leq 1 \\ -\varpi/2 \leq u \leq \varpi/2 \end{array} \right)$$

とする．

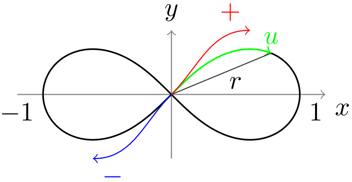
三角関数のときと同様に $r^2 = \cos 2\theta$ の微分を考える．$r dr = -\sin 2\theta \, d\theta$ より，

$$\frac{d\theta}{dr} = -\frac{r}{\sin 2\theta} = -\frac{r}{\sqrt{1-r^4}}$$

$$\Rightarrow u = \int_0^r \sqrt{1 + \left(r \frac{d\theta}{dr} \right)^2} dr$$

$$= \int_0^r \sqrt{\frac{1 - r^4 + r^4}{1 - r^4}} dr$$

$$= \int_0^r \frac{dr}{\sqrt{1-r^4}}.$$



$\rightsquigarrow$ 幾何学的に $\mathbb{R}$ 全体に拡張する．

三角関数のときと同様に考えて

Proposition.

$$\begin{aligned} \mathrm{sl}(u+2\varpi) &= \mathrm{sl}\,u, \\ \mathrm{sl}(-u) &= -\mathrm{sl}\,u. \end{aligned}$$

Theorem.

u, v を十分小さくとれば次式が成立；

$$\mathrm{sl}(u+v) = \frac{\mathrm{sl}\,u\sqrt{1-\mathrm{sl}^4\,v} + \mathrm{sl}\,v\sqrt{1-\mathrm{sl}^4\,u}}{1+\mathrm{sl}^2\,u\,\mathrm{sl}^2\,v}.$$

が成立する.

cl 関数を次のようにして定める.

Definition.

$$\operatorname{cl} u := \operatorname{sl}\left(\frac{\varpi}{2} - u\right).$$

Proposition.

$$\operatorname{cl}^2 u + \operatorname{cl}^2 u \operatorname{sl}^2 u + \operatorname{sl}^2 u = 1.$$

Theorem.

$$\operatorname{sl}(u+v) = \frac{\operatorname{sl} u \operatorname{cl} v + \operatorname{cl} u \operatorname{sl} v}{1 - \operatorname{sl} u \operatorname{cl} u \operatorname{sl} v \operatorname{cl} v}.$$

Question.

$\operatorname{sl} \dfrac{\varpi}{6}$ を求めよ.

Definition.

$v \in \mathbb{R}$, $i = \sqrt{-1}$ とする.

$$\operatorname{sl} iv := i \operatorname{sl} v, \quad \operatorname{cl} iv := \frac{1}{\operatorname{cl} v}.$$

$\because$

$$\int_0^{iy_0} \frac{dx}{\sqrt{1-x^4}} \underset{(x=iy)}{=} i \int_0^{y_0} \frac{dy}{\sqrt{1-y^4}} =: iv$$

より

$$\operatorname{sl} iv = iy_0 = i \operatorname{sl} v,$$

$$\operatorname{cl} iv = \sqrt{\frac{1-\operatorname{sl}^2 iv}{1+\operatorname{sl}^2 iv}} = \sqrt{\frac{1+\operatorname{sl}^2 v}{1-\operatorname{sl}^2 v}} = \frac{1}{\operatorname{cl} v}.$$

加法定理とあわせて，

Definition.

$u, v \in \mathbb{R}$ とする．

$$\mathrm{sl}(u+iv) := \frac{\mathrm{sl}\,u\,\mathrm{cl}\,iv + \mathrm{cl}\,u\,\mathrm{sl}\,iv}{1 - \mathrm{sl}\,u\,\mathrm{cl}\,u\,\mathrm{sl}\,iv\,\mathrm{cl}\,iv}.$$

Proposition.

$$\mathrm{sl}(u+2\varpi i) = \mathrm{sl}\,u \quad (\forall u \in \mathbb{C}).$$

Proof.

$u = a + bi$, $(a, b \in \mathbb{R})$ とおけば

$$\begin{aligned}\mathrm{sl}(u+2\varpi i) &= \mathrm{sl}\left(a + (b+2\varpi)i\right) \\ &= \mathrm{sl}(a+bi) = \mathrm{sl}\,u.\end{aligned}$$

□

Definition.

$\mathbb{R}$ 上独立な 2 つの周期を持つ有理型関数を楕円関数という.

Remark.

sl の基本周期は実は 2ϖ と $\varpi + \varpi i$ である.

Definition.

$a, b > 0$, $a_0 := a$, $b_0 := b$

$$a_{n+1} := \frac{a_n + b_n}{2}, \quad b_{n+1} := \sqrt{a_n b_n}$$

とする．このとき，

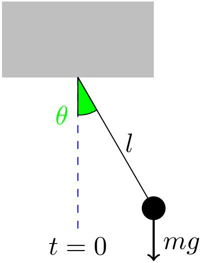
$$M(a,b) := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

と定め，a と b の "算術幾何平均" とよぶ．

Theorem.

$$M(1, \sqrt{2}) = \frac{\pi}{\varpi}.$$

単振り子



$$E = \frac{1}{2}m\left(l\dot{\theta}\right)^2 + mgl\left(1-\cos\theta\right)$$
$$= mgl\left(1-\cos\theta_0\right)$$

(ただし $-\theta_0 \leq \theta \leq \theta_0$.)

ここで $\boxed{m = g = l = 1}$ とする.

$$\frac{1}{2}\dot{\theta}^2 + (1 - \cos\theta) = 1 - \cos\theta_0$$

$$\Rightarrow \frac{1}{2}\dot{\theta}^2 + 2\sin^2\frac{\theta}{2} = 2\sin^2\frac{\theta_0}{2}$$

$$\Rightarrow \dot{\theta}^2 = 4\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right)$$

$$= 4k^2\left(1 - \sin^2\varphi\right)$$

$$= 4k^2\cos^2\varphi$$

$$\therefore \frac{d\theta}{dt} = 2k\cos\varphi$$

$$k := \sin\frac{\theta_0}{2}$$

$$k\sin\varphi := \sin\frac{\theta}{2}$$

$$k\cos\varphi\, d\varphi = \frac{1}{2}\cos\frac{\theta}{2}d\theta$$

$$= \frac{1}{2}\sqrt{1 - k^2\sin^2\varphi}\, d\theta$$

$$d\theta = \frac{2k\cos\varphi\, d\varphi}{\sqrt{1 - k^2\sin^2\varphi}}$$

$$\begin{aligned} t = \int_0^t dt &= \int_0^\theta \frac{d\theta}{2k\cos\varphi} \\ &= \int_0^\varphi \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} \\ &= \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \end{aligned}$$

$$\begin{aligned} x &= \sin\varphi \\ dx &= \cos\varphi\, d\varphi \\ &= \sqrt{1-x^2}\, d\varphi \end{aligned}$$

$$\begin{array}{c|ccc} \theta & 0 & \to & \theta_0 \\ \varphi & 0 & \to & \pi/2 \\ x & 0 & \to & 1 \end{array}$$

Jacobi の楕円関数

Definition.

$$K(k) = K := \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

と定める．ここで
K：第一種完全楕円積分，k：母数，$k' := \sqrt{1-k^2}$：補母数 という．

Definition.

$$x = \operatorname{sn}(u,k) = \operatorname{sn} u \overset{\text{def.}}{\Longleftrightarrow} u = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

また，

$$\operatorname{cn} u := \sqrt{1 - \operatorname{sn}^2 u},$$
$$\operatorname{dn} u := \sqrt{1 - k^2 \operatorname{sn}^2 u}.$$

※今までと同様に $\mathbb{R}$ 全体に拡張しておく．

定義からすぐ分かること

$$
\begin{array}{c|c}
k \to 0 & \operatorname{sn} u \to \sin u, \\
k = i & \operatorname{sn} u = \operatorname{sl} u, \qquad \operatorname{cl} u = \operatorname{cn} u / \operatorname{dn} u \\
k \to 1 & \operatorname{sn} u \to \tanh u
\end{array}
$$

Proposition.

$$
\begin{aligned}
\operatorname{sn}(-u) &= -\operatorname{sn} u, & \operatorname{sn}(u+4K) &= \operatorname{sn} u, \\
\operatorname{cn}(-u) &= \operatorname{cn} u, & \operatorname{cn}(u+4K) &= \operatorname{cn} u, \\
\operatorname{dn}(-u) &= \operatorname{dn} u, & \operatorname{dn}(u+2K) &= \operatorname{dn} u.
\end{aligned}
$$

u	0	K	$2K$	$3K$	$4K$
$\operatorname{dn} u$	1	k'	1	k'	1

Theorem. (sn の加法定理)

$$\mathrm{sn}(u+v) = \frac{\mathrm{sn}\,u\,\mathrm{cn}\,v\,\mathrm{dn}\,v + \mathrm{sn}\,v\,\mathrm{cn}\,u\,\mathrm{dn}\,u}{1 - k^2\,\mathrm{sn}^2\,u\,\mathrm{sn}^2\,v}$$

Theorem. (Landen 変換)

$k_1 := \dfrac{1-k'}{1+k'}$ とおく．このとき

$$\mathrm{sn}\left((1+k')u, k_1\right) = (1+k')\frac{\mathrm{sn}(u,k)\,\mathrm{cn}(u,k)}{\mathrm{dn}(u,k)}$$

が成立.

Proposition.

$$K = \frac{2}{1+k'} K_1 \quad \bigl(K_1 := K(k_1)\bigr)$$

Proof.

Landen 変換で $u = K$ とすると

$$\begin{aligned} \operatorname{sn}\bigl((1+k')K, k_1\bigr) &= (1+k') \frac{\operatorname{sn}(K,k)\operatorname{cn}(K,k)}{\operatorname{dn}(K,k)} \\ &= 0 \quad (\because \operatorname{cn}(K,k) = 0, \operatorname{dn}(K,k) \neq 0). \end{aligned}$$

よって $(1+k')K = 2nK$ であるが，実は $n = 1$. □

Theorem.

$a, b > 0,\ k' := b/a$ とする．このとき

$$M(1, k') = \frac{\pi}{2K}.$$

Proof.

$k'_1 := \sqrt{1 - k_1^2}$ とおけば，

$$k'_1 = \sqrt{1 - \left(\frac{1-k'}{1+k'}\right)^2} = \sqrt{\frac{(1+k')^2 - (1-k')^2}{(1+k')^2}}$$
$$= \sqrt{\frac{4k'}{(1+k')^2}} = \frac{2\sqrt{k'}}{1+k'}.$$

Proof (Cont.)

つまり

$$k_1' = \frac{2\sqrt{k'}}{1+k'}.$$

ここで $k' = b/a$ としたので

$$\begin{aligned} k_1' &= \frac{2\sqrt{b/a}}{1+(b/a)} \\ &= \frac{2}{a+b}\sqrt{ab} \\ &= \frac{b_1}{a_1}. \end{aligned}$$

Proof (Cont.)

ここで $I(a,b) := \int_0^{\pi/2} \frac{d\varphi}{\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}}$ と書くと

$$
\begin{aligned}
K &= \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} \\
&= \int_0^{\pi/2} \frac{d\varphi}{\sqrt{\cos^2\varphi + (1-k^2)\sin^2\varphi}} \\
&= I(1,k') = I\left(1, \frac{b}{a}\right) \\
&= aI(a,b)
\end{aligned}
$$

同様に $K_1 = a_1 I(a_1, b_1)$ が分かる.

Proof (Cont.)

Landen 変換の後の Proposition より

$$K = \frac{2}{1+k'}K_1 = \frac{2}{1+(b/a)}K_1$$
$$= \frac{2a}{a+b}K_1 = \frac{a}{a_1}K_1.$$

よって

$$aI(a,b) = \frac{a}{a_1}a_1 I(a_1,b_1) \quad \therefore I(a,b) = I(a_1,b_1).$$

同様にして

$$\frac{K}{a} = I(a,b) = I(a_1,b_1) = I(a_2,b_2) = \cdots = I(a_\infty,b_\infty).$$

Proof (Cont.)

$I(a_\infty, b_\infty) = \frac{1}{M(a,b)} I(1,1)$ であるから

$$\begin{aligned}\frac{K}{a} &= I(a_\infty, b_\infty) = \frac{1}{M(a,b)} I(1,1) \\ &= \frac{1}{aM(1,b/a)} \int_0^{\pi/2} d\varphi \\ &= \frac{1}{aM(1,k')} \frac{\pi}{2} \qquad \therefore M(1,k') = \frac{\pi}{2K}\end{aligned}$$

□

Corollary.

$$M(1,\sqrt{2}) = \frac{\pi}{\varpi}$$

Definition.

$v \in \mathbb{R}$, $i = \sqrt{-1}$ とする.

$$\mathrm{sn}(iv, k) := i\frac{\mathrm{sn}(v, k')}{\mathrm{cn}(v, k')},$$

$$\mathrm{cn}(iv, k) := \frac{1}{\mathrm{cn}(v, k')},$$

$$\mathrm{dn}(iv, k) := \frac{\mathrm{dn}(v, k')}{\mathrm{cn}(v, k')}.$$

これは $y_0 \in \mathbb{R}$ をとって

$$\int_0^{iy_0} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

を考えればよい.

$u, v \in \mathbb{R}$ に対し $\mathrm{sn}(u+iv,k)$ 等を加法定理の式で定めれば

	周期	極（1 位）	零点（1 位）
sn	$4K, 2K'i$	$2mK+(2n-1)K'i$	$2mK+2nK'i$
cn	$4K, 2K+2K'i$	〃	$(2m-1)K+2nK'i$
dn	$2K, 4K'i$	〃	$(2m-1)K+(2n-1)K'i$

ただし $K' := K(k')$, $m, n \in \mathbb{Z}$ である.

テータ関数

簡単のため次の記法を用いる；

$$\prod := \prod_{n=1}^{\infty}, \sum := \sum_{n=-\infty}^{\infty}$$

Definition.

$v \in \mathbb{C}$, $\tau \in \mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$, $z := e^{\pi i v}$, $q := e^{\pi i \tau}$ とおく．

$$\vartheta_1(v) := Cq^{1/4} \frac{z - z^{-1}}{i} \prod (1 - q^{2n} z^2)(1 - q^{2n} z^{-2})$$

$$\vartheta_2(v) := Cq^{1/4} (z + z^{-1}) \prod (1 + q^{2n} z^2)(1 + q^{2n} z^{-2})$$

$$\vartheta_3(v) := C \prod (1 + q^{2n-1} z^2)(1 + q^{2n-1} z^{-2})$$

$$\vartheta_0(v) := C \prod (1 - q^{2n-1} z^2)(1 - q^{2n-1} z^{-2})$$

ここで $C := \prod (1 - q^{2n})$ である．また $\vartheta_3 := \vartheta_3(0)$ 等と書く．

ここで

$$
\begin{aligned}
\vartheta_0(v) = 0 &\Leftrightarrow q^{2n-1} z^{\pm 2} = 1 \\
&\Leftrightarrow e^{\pi i \left((2n-1)\tau \pm 2v \right)} = 1 \\
&\Leftrightarrow (2n-1)\tau \pm 2v = 2m \\
&\Leftrightarrow v = \pm m \mp (n - \frac{1}{2})\tau
\end{aligned}
$$

	零点	$\tau = \frac{K'}{K}i,\ v = \frac{u}{2K}$ のとき
ϑ_1	$m + n\tau$	$2mK + 2nK'i$
ϑ_2	$\left(m - \frac{1}{2}\right) + n\tau$	$(2m-1)K + 2nK'i$
ϑ_3	$\left(m - \frac{1}{2}\right) + \left(n - \frac{1}{2}\right)\tau$	$(2m-1)K + (2n-1)K'i$
ϑ_0	$m + \left(n - \frac{1}{2}\right)\tau$	$2mK + (2n-1)K'i$

$$\vartheta_0 = 0\left(\frac{u}{2K}\right) \Leftrightarrow \frac{u}{2K} = m + \left(n - \frac{1}{2}\right)\frac{K'}{K}i$$

$$u = 2mK + (2n-1)K'i$$

Proposition.

$$\operatorname{sn} u = C_1 \frac{\vartheta_1(v)}{\vartheta_0(v)},$$
$$\operatorname{cn} u = C_2 \frac{\vartheta_2(v)}{\vartheta_0(v)},$$
$$\operatorname{dn} u = C_3 \frac{\vartheta_3(v)}{\vartheta_0(v)}.$$

Proposition.

$$\vartheta_k(v+2) = \vartheta_k(v) \quad (k = 1, 2, 3, 0)$$

Proof.

$v \to v+2$ のとき， $e^{\pi i(v+2)} = e^{\pi i v}$ より $z \to z$ から分かる． □

Proposition. (ϑ_k の Fourier 展開)

$$\vartheta_1(v) = i\sum(-1)^n q^{\left(\frac{2n-1}{2}\right)^2} z^{2n-1}$$

$$\vartheta_2(v) = \sum q^{\left(\frac{2n-1}{2}\right)^2} z^{2n-1}$$

$$\vartheta_3(v) = \sum q^{n^2} z^{2n}$$

$$\vartheta_0(v) = \sum(-1)^n q^{n^2} z^{2n}$$

Proposition. (熱方程式)

$$\frac{\partial^2 \vartheta_k}{\partial v^2} = 4\pi i \frac{\partial \vartheta_k}{\partial \tau}$$

Proof.

項別微分すればよい. □

Fact.

$$\wp(u) - \wp(v) = -\vartheta_1'^2 \frac{\vartheta_1(u+v)\vartheta_1(u-v)}{\vartheta_1(u)^2\vartheta_1(v)^2}.$$

ただし $\wp$ の周期は $1, \tau$ である.

Proposition.

$$\vartheta_3^4 = 4q \frac{d}{dq} \log \frac{\vartheta_2}{\vartheta_0}.$$

Theorem.

$n \in \mathbb{N}$, $\Phi(n) := \#\{(n_1, n_2, n_3, n_4) \in \mathbb{Z}^4 \mid n_1^2 + n_2^2 + n_3^2 + n_4^2 = n\}$ とする．このとき

$$\sigma_1(n) := \sum_{\substack{k \in \mathbb{N} \\ k | n}} k,$$

$$\sigma_2(n) := \sum_{\substack{k \in 4\mathbb{N} \\ k | n}} k,$$

$$\sigma(n) := \sigma_1(n) - \sigma_2(n)$$

とすれば

$$\Phi(n) = 8\sigma(n).$$

Proof of Theorem.

$\vartheta_3 = \sum q^{n^2}$ より $\vartheta_3^4 = \sum_{n=0}^{\infty} \Phi(n) q^n$ が分かる．よって

$$\begin{aligned}\frac{\vartheta_2}{\vartheta_0} &= \frac{2q^{\frac{1}{4}} \prod(1+q^{2n})^2(1-q^{2n})^2}{\prod(1-q^{2n-1})^2(1-q^{2n})^2} \\ &= 2q^{\frac{1}{4}} \frac{\prod(1-q^{4n})^2}{\prod(1-q^n)^2}\end{aligned}$$

この自然対数をとって

$$\begin{aligned}\log\frac{\vartheta_2}{\vartheta_0} &= \log 2 + \frac{1}{4}\log q + 2\sum_{n=1}^{\infty}\log(1-q^{4n}) + \sum_{n=1}^{\infty}\log(1-q^n) \\ \Rightarrow \frac{d}{dq}\log\frac{\vartheta_2}{\vartheta_0} &= \frac{1}{4q} + 2\sum_{n=1}^{\infty}\frac{-4nq^{4n-1}}{1-q^{4n}} - 2\sum_{n=1}^{\infty}\frac{-nq^{n-1}}{1-q^n}\end{aligned}$$

Proof (Cont.)

$$
\begin{aligned}
\vartheta_3^4 &= 1 - 8\sum_{n=1}^{\infty} \frac{4nq^{4n}}{1-q^{4n}} + 8\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} \\
&= 1 - 8\sum_{n=1}^{\infty} 4nq^{4n} \sum_{m=0}^{\infty} q^{4nm} + 8\sum_{n=1}^{\infty} nq^n \sum_{m=0}^{\infty} q^{nm} \\
&= 1 - 8\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} 4nq^{4n(m+1)} + 8\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} nq^{n(m+1)} \\
&= 1 - 8\sum_{n=1}^{\infty} \sigma_2(n)q^n + 8\sum_{n=1}^{\infty} \sigma_1(n)q^n \\
&= 1 + 8\sum_{n=1}^{\infty} \sigma(n)q^n
\end{aligned}
$$

以上より $\Phi(n) = 8\sigma(n)$. □